# Fostering mathematical understanding through physical and virtual manipulatives

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When solving mathematical problems, many students know the procedure to get to the answer but cannot explain why they are doing it in that way. According to Skemp (1976) these students have instrumental understanding but not relational understanding of the problem. They have accepted the rules to arriving at the answer without questioning or understanding the underlying reasons for why a certain procedure is carried out. To help students grasp abstract mathematical concepts and form relational understanding of these concepts, research has found that it is often necessary to make use of physical or virtual materials to help scaffold their understanding and/or simplify the abstract idea (Sowell, 1989; Suh & Moyer, 2008). This paper presents some ways in which fundamental concepts such as subtraction with regrouping, equivalent fractions, dividing and multiplying fractions, and measurement topics such as area and perimeter, can be explored and clarified. A range of physical and virtual manipulatives are suggested to help foster and consolidate the relational understanding needed to grasp these concepts. A number of examples are provided which are suitable for teachers from primary through to middle years. Even though some of these concepts seem basic and related to primary mathematics, they are addressed here because they underpin the efficient working out of the more abstract concepts associated with middle school mathematics. Having a strong relational understanding and subsequent mastery of these concepts help prevent misconceptions and errors, and position students better in their mathematics learning. Additionally, these activities and strategies have the potential to help struggling middle school students grasp these basic concepts.

Physical manipulatives aid deep conceptual understanding because they present alternative representations that help reconstruct concepts (Shakow, 2007 cited in Yuan, 2009) and aid concrete thinking (Sowell, 1989). Students who have worked with manipulatives tend to perform better in maths (Raphael & Wahlstorm, 1989). Terry (1995) found that a combination of concrete and virtual manipulatives helped students make significant gains compared to students using only physical manipulatives or only virtual manipulatives. Takahashi (2002 cited in Moyer, Salkind & Bolyard, 2008) similarly noted that students benefitted from instruction from both physical and virtual geoboards. Linked representations in the virtual fraction environment have been found to offer meta-cognitive support by keeping record of the user's

actions and numeric notations. This support allowed special needs children working with equivalent fraction to observe and reflect on connections and relationships among the representations (Suh & Moyer, 2008). Special needs children (aged 8–12) with difficulty in subtraction problems up to 100 with the ones-digit subtrahend being larger than the ones digit in the minuend have benefitted from using dynamic virtual manipulative (Peltenburg, van den-Heuvel-Panhuizen & Doig, 2009).

## Selecting manipulatives

When selecting manipulatives, Zbiek, Heid, Blume & Dick (2007) recommended that the following aspects be considered.

- **Mathematical fidelity**: the degree to which the mathematical object is faithful to the underlying mathematical properties of that object in the virtual environment.
- **Cognitive fidelity**: how well the virtual tool reflect the user's cognitive actions and possible choices while using the tool in the virtual environment
- **Pedagogical fidelity**: the extent to which teachers and students believe that a tool allows students to act mathematically in ways that correspond to the nature of mathematical learning that underlies a teacher's practice.

The following section looks at how careful selection of physical and virtual manipulatives can potentially help address common errors and misconceptions.

## Addition and subtraction requiring regrouping

One of the most common errors students make with number operations relate to the subtraction of one number from the other where one of the digits in the subtrahend is larger than the corresponding digit in the minuend (see Young & Shea, 1981). See example in the problem below.

$$\begin{array}{rrrrrl} & 1 & 2 & 8 & 3 & \text{(minuend)} \\ - & & 1 & 6 & 7 & \text{(subtrahend)} \\ \hline & 1 & 1 & 2 & 4 & \end{array}$$

This problem shows that the digit in the ones place value of the subtrahend (7) is larger than the corresponding digit (3) in the minuend. Students often mistakenly subtract the 3 from 7 as they are often told to subtract the smaller digit from the larger digit resulting in the answer being 1124 instead of 1116. Teachers often find that students become even more confused when zero is one of the digits in the minuend; e.g., 1203 (minuend) – 167 (subtrahend). What can be done to help these students overcome these problems?

## Understanding place value

To help students overcome such problems, it is instructive to help them understand the place value of each digit in a number. By using base-10 blocks and a place value mat (see Figure 1) and other concrete manipulatives such as number expanders or arrow cards, students know the value for each digit in a number.

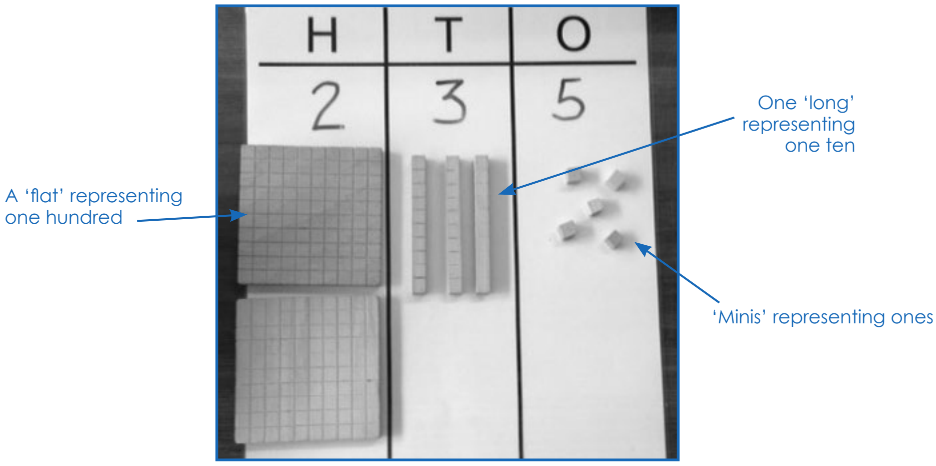


Figure 1. Place value mat and Base 10 blocks

This understanding can be consolidated through the use of a virtual manipulative such as the virtual base 10 blocks in the website National Library of Virtual Manipulatives (NLVM). In this applet, the number of decimal places on the applet can be changed to enable students to learn about place value for whole numbers and decimals. This applet can also be used to teach bases other than base 10. The number of columns in the place value chart can also be varied from two to four thereby allowing for up to four digit numbers. Instructions and explanations are provided to the user on how to use the applet (see Figure 2).

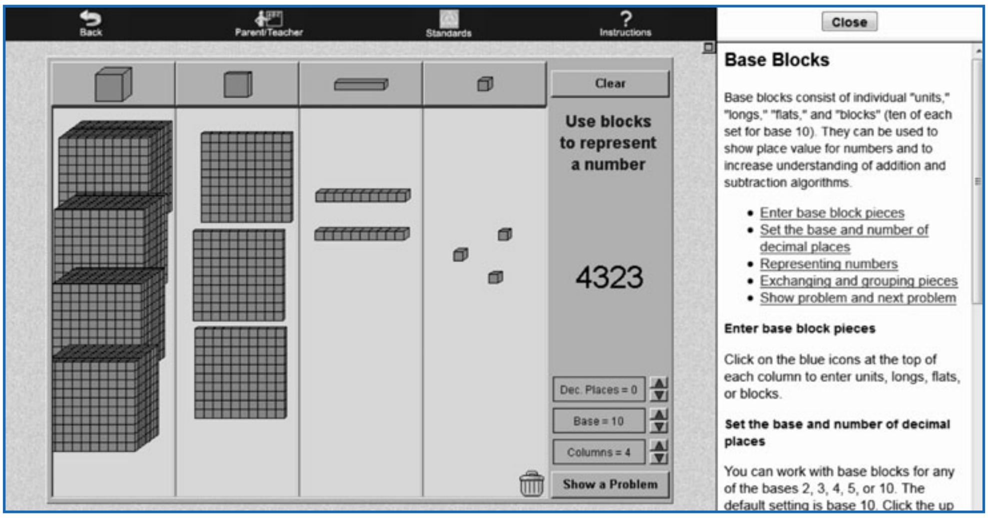


Figure 2. Base Blocks
Source: http://nlvm.usu.edu/en/nav/frames_asid_152_g_1_t_1.html?from=grade_g_1.html

Once students are familiar with place value concepts, addition and subtraction of numbers can be carried out. To assist students who have difficulty working out addition problems requiring trading, base 10 blocks can be used. By playing the game 'Win a flat' where students roll a die to determine the number of units to be added and when the units added together becomes more than ten, they can be substituted with a higher unit that represents ten, trading or regrouping is carried out and the person who is first to trade for a flat wins the game. This concept can also be reinforced using the Base 10 Addition blocks in the NLVM site. Likewise subtraction concepts can be reinforced by playing the game "Lose a flat" or using virtual Base 10 subtraction blocks. By dragging a tens bar to the ones column, the bar splits into ones which allows subtraction to occur (see Figure 3).

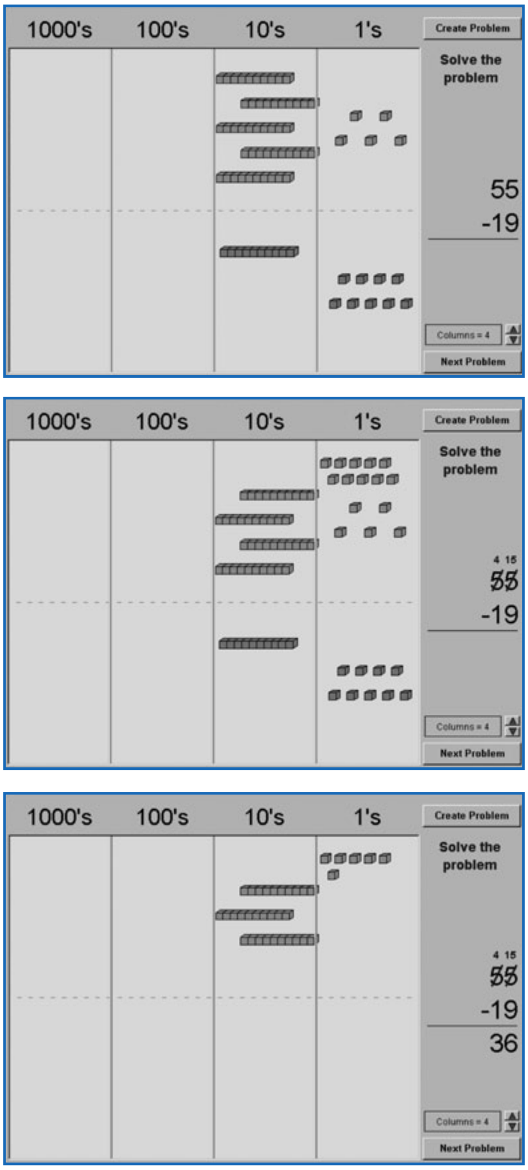


Figure 3. Base blocks subtraction Source: http://nlvm.usu.edu/en/nav/frames_asid_155_g_3_t_1.html?from=category_g_3_t_1.html

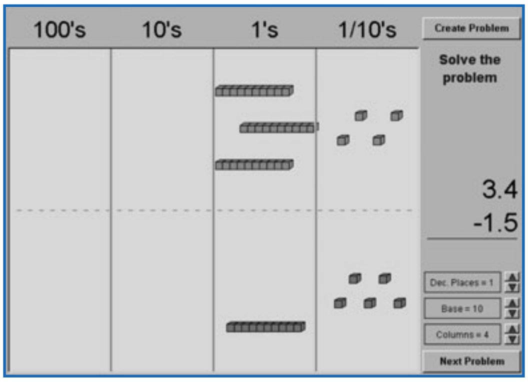


Figure 4. Base blocks decimals. Source: http://nlvm.usu.edu/en/nav/frames_asid_264_g_3_t_1.html?from=category_g_3_t_1.html

Base block decimals (Figure 4) can also be used to explore and reinforce the idea of subtraction of decimals by replacing the idea that the unit block is one tenth.

The physical and virtual imagery of a lesser number of units in the minuend as well as the inbuilt mechanism in the software that inherently disallow the subtraction of the larger unit in the subtrahend from the minuend means that students have to rethink their steps so that regrouping must first occur before subtraction can take place. This is a form of scaffolding that enables students to remember to regroup or trade in subtraction or addition problems.

## Fractions

### Naming fractions

When students work with fractions, it is not uncommon to see problems such as the one in Figure 5 answered in this manner. To represent one third and one sixth, the student has divided the circle into three parts and six parts respectively without realising that each of the parts have to be equal in size and shape. This might be because the student is familiar with dividing a square or a rectangle into equal parts by drawing lines of equal widths in a square or a rectangle.

To ensure a comprehensive understanding of fractions that is not limiting it is advisable to use a variety of whole shapes such as squares, rectangles, circles and triangles and fraction parts as well as sets (see Figure 6). It is important to highlight that when comparing fractions, the whole needs to be the same (Figure 7).

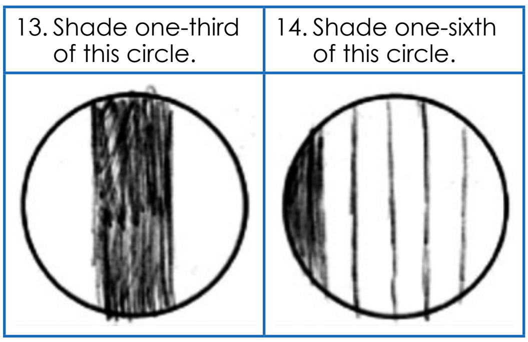


Figure 5. Misconceptions in fractions (Source: Gould, P. (2005, p. 9))

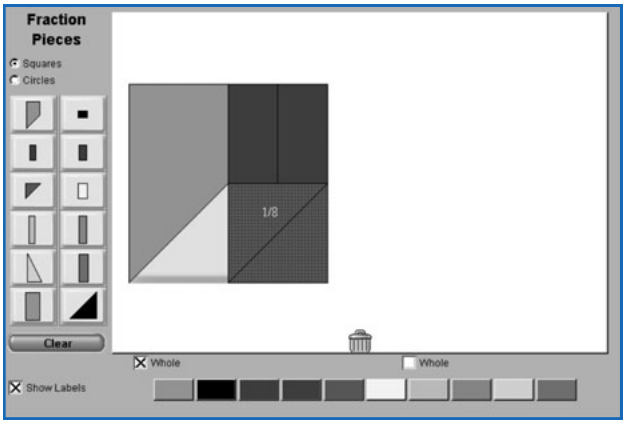


Figure 6. Identifying fractional parts in a whole.

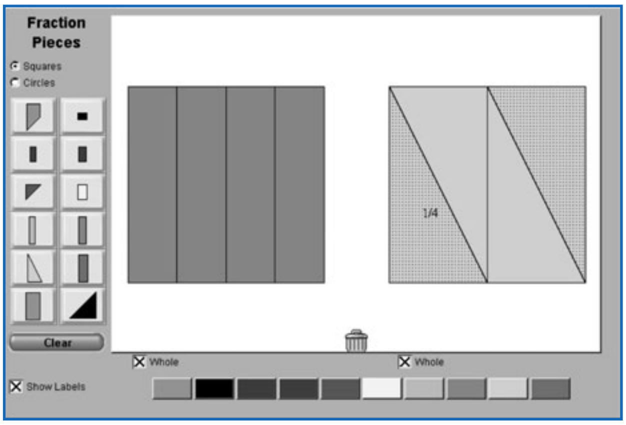


Figure 7. Different representations of the same fraction (NLVM).

## Addition of fractions

In spoken form, $\frac{1}{5} + \frac{1}{5}$ is often read as one fifth plus one fifth is two fifths. However, a common error students make when expressing this in written symbolic form is

$$\frac{1}{5}+\frac{1}{5}=\frac{2}{10}$$

This is because the numerators and denominators are regarded and treated as whole numbers instead of being parts of a fraction. Thus,

$$\frac{1}{3}+\frac{1}{5}=\frac{2}{8}$$

for example, is a common error that needs to be addressed. Adding fractions with different denominators can present problems for some students. The root cause of this problem lies in students not understanding equivalent fractions. To help students learn about equivalent fractions, paper strips can be used to make equivalent fractions to form a fraction wall. Each strip shows a fraction and by placing them side by side, students can match the fractions that fit into another. For example one half is the equivalent to two quarters and one third is equivalent to two sixths. It is important to highlight that in order to compare fractions, the whole must be of the same shape and size.

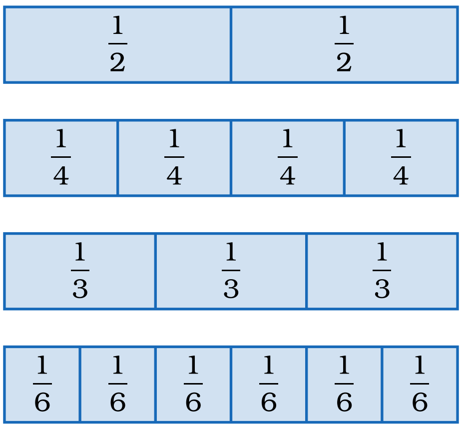


Figure 8. Fraction strips.

To reinforce this idea and to give students a range of values for the denominator, virtual fraction strips can be used and can be found in this applet (see Figure 9).

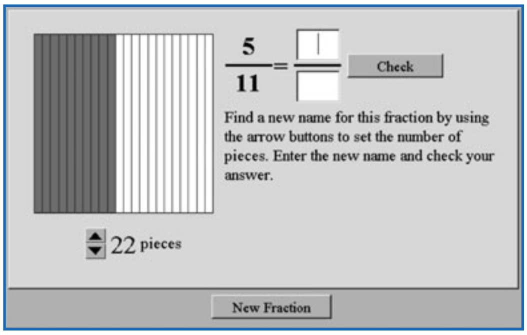


Figure 9. Equivalent fractions. Source: http://nlvm.usu.edu/en/nav/frames_asid_105_g_3_t_1.html?from=category_g_3_t_1.html

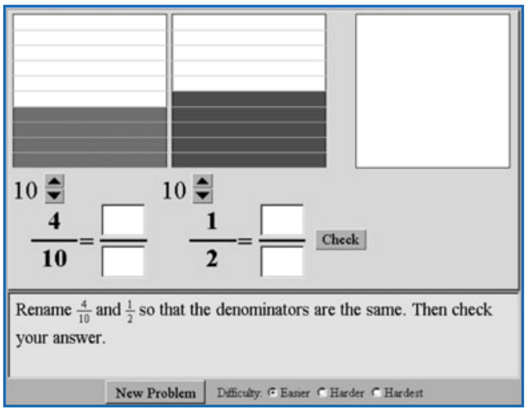


Figure 10. Renaming fractions into equivalent fractions with the same denominator.

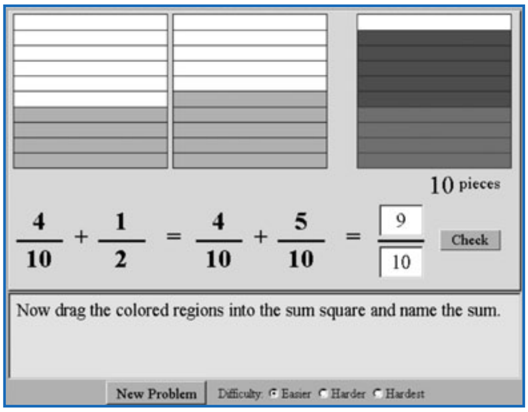


Figure 11. Adding fractions with the same denominators. Source: http://nlvm.usu.edu/en/nav/frames_asid_106_g_3_t_1.html?from=category_g_3_t_1.html

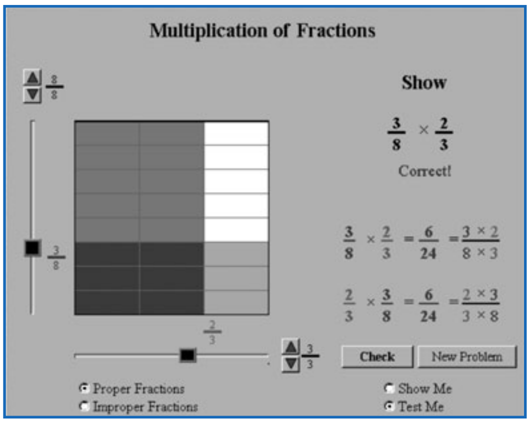


Figure 13. Applet showing multiplication of fractions. Source: http://nlvm.usu.edu/en/nav/frames_asid_194_g_3_t_1.html?from=category_g_3_t_1.html

Through this virtual manipulative students can learn to rename fractions so that they have the same denominator before adding the fractions together (see Figures 10 and 11).

## Multiplication and division of fractions

Conceptual understanding of the multiplication and division of fractions is also problematic for some students. However, by using paper strips and suitable language the multiplication of fractions can be made more explicit. For example,

$$\frac{1}{3}\times\frac{1}{2}$$

can be read as 'one third of one half'.

By folding the strip into two we get a half. Folding the half into thirds gives six parts in the whole when we open up the paper strip. Hence one third of one half is one sixth (Figure 12).

| | |
|---|---|
| $\frac{1}{2}$ | $\frac{1}{2}$ |

| | | | | | |
|---|---|---|---|---|---|
| $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

Figure 12

When the denominators of fractions become larger it is difficult to manipulate the folding of paper strips. A virtual manipulative then becomes more manageable and visually compelling. Figure 13 shows what multiplication of two fractions means in pictorial form.

Another common problem that confronts students is the division of fractions. A fraction divided by a whole number is quite easily done using a paper strip. For example, $\frac{1}{2}$ divided by 3 means half of the strip is divided into three equal parts and each part becomes $\frac{1}{6}$. However when a fraction is divided by a fraction, using a virtual fraction bar allows this concept to be made clearer. Hence $\frac{3}{4}$ divided by $\frac{1}{2}$ is $1\frac{1}{2}$ times of halves (see Figure 14).

## Misconceptions in area and perimeter

Area and perimeter are terms that are often confused or used interchangeably by students and the units are often wrongly attributed; e.g., perimeter incorrectly given in square centimetres. A common misconception in students as well as adults is the same-perimeter/same-area misconception (Dembo, Levin & Siegler, 1997). Shapes with the same perimeter are thought of as having the same area. A manipulative that can help students overcome this problem is the geoboard. The physical geoboard is either a board with nails or pegs lined up in rows and columns (Figure 15). By counting the number of squares or the length of the sides, area and perimeter can be differentiated. This tool is also ideal for exploring how the area of a shape change with perimeter. Students will learn that the more regular the shape is the larger the area. Hence the closer the rectangle approximates to a square the larger the area and conversely the closer the rectangle approaches a line the greater the decrease in area. While these geoboards are versatile and effective tools, they can be cumbersome to carry around.

Figure 16 shows a virtual geoboard which is similar in function, colourful and easily managed

By creating various shapes with the bands and exploring the area and perimeter of each of the shapes, students can be asked to investigate the relationship between area and perimeter. How does perimeter change if the area is kept constant? This kind of concrete albeit virtual manipulations is supported by the basic intuition that an area is proportional to the number of units contained within it (Dembo et al., 1997). Other areas of investigations can include: How does area increase or decrease with the same perimeter (e.g., when the points of the shape are as far apart as possible)? What is the maximum area that can be created while keeping the perimeter to a minimum? This manipulative provides concrete evidence that allows students to see the consequences of changing the shape through the 'Measures' feature and to see that increases in perimeter does not necessarily result in increase in area. Students' reasoning skills can potentially be sharpened as they justify and explain their conjectures.

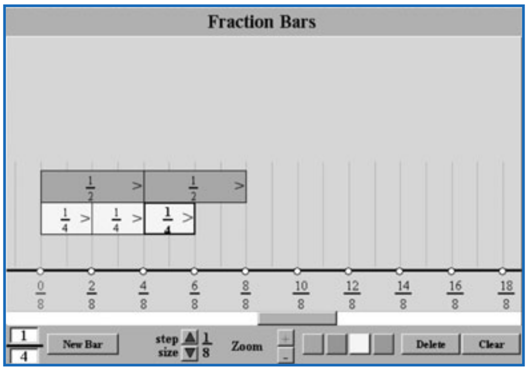


Figure 14. Number line bars showing fractions. Source: http://nlvm.usu.edu/en/nav/frames_asid_106_g_3_t_1.html?from=category_g_3_t_1.html

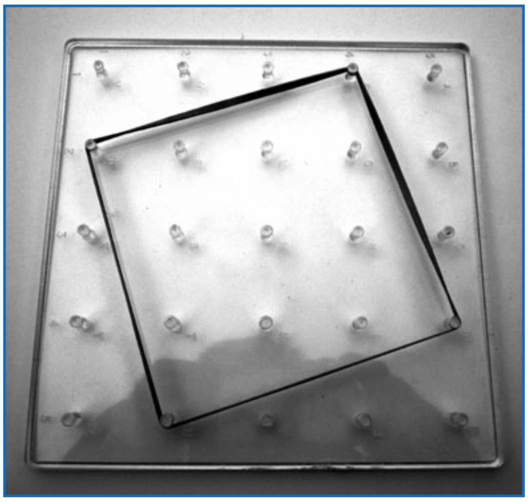

Figure 15. Geoboard. Source: http://commons.wikimedia.org/wiki/File:25_peg_geoboard.JPG

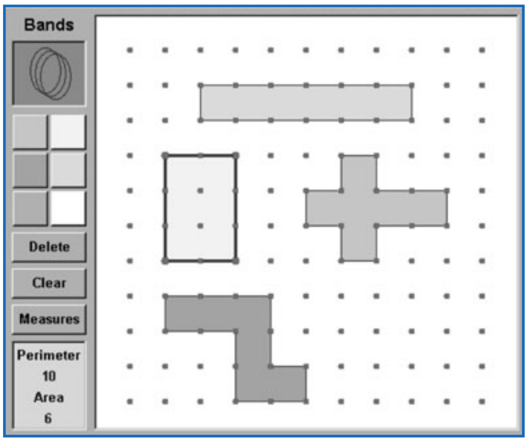


Figure 16. Virtual geoboard. Source: http://nlvm.usu.edu/en/nav/frames_asid_106_g_3_t_1.html?from=category_g_3_t_1.html

## Conclusion

This paper has presented different ways in which physical and virtual manipulatives can be used to assist students make meaning of fundamental yet seemingly problematic mathematical concepts such as place value and regrouping, multiplying and dividing fractions and, area and perimeter. Whilst the time spent investigating and exploring these concepts can be considerable, it will be time well invested as potential misconceptions can be alleviated as these manipulatives help students appreciate and grasp the concepts better. These activities may also be used as remedial measures for struggling students whose progress into middle school mathematics are impeded due to poor relational understanding of these concepts. For an abstract or symbolic idea to have meaning for these students, it is sometimes necessary to provide a connection using some form of concrete, kinaesthetic and/or visual experience so that an 'aha!' moment can occur. It is imperative that as teachers we facilitate such moments.

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